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Stability of inviscid shear flow over a flexible boundary

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The stability of an inviscid flow that comprises a thin shear layer and a uniform outer flow over a flexible boundary is investigated. It is shown that the flow is temporally unstable for all wavenumbers. This instability is either Kelvin–Helmholtz-like or induced by the phase shift across the critical layer. The threshold of absolute instability is determined in the form $\boldsymbol{F} = \boldsymbol{F}_*(1 + \boldsymbol{C}\varepsilon^{\boldsymbol{n}})$ for $\varepsilon \ll 1$, where $\boldsymbol{F}$ (a Froude number) and ε are, respectively, dimensionless measures of the flow speed and the shear-layer thickness, $\boldsymbol{F}_*$ is the limiting value of $\boldsymbol{F}$ for a uniform flow, $\boldsymbol{C} < 0$ and $\boldsymbol{n} = 1$ in the absence (as for a broken-line velocity profile) of a phase shift across the critical layer, and $\boldsymbol{C} > 0$ and $\boldsymbol{n} = 2/3$ in the presence of such a phase shift. Explicit results are determined for an elastic plate (and, in an Appendix, for a membrane) with a broken-line, parabolic, or Blasius boundary-layer profile. The predicted threshold for the broken-line profile agrees with Lingwood & Peake's (1999) result for $\varepsilon \ll 1$, but that for the Blasius profile contradicts their conclusion that the threshold for $\varepsilon \downarrow 0$ is a 'singular and unattainable limit'.

1. Introduction

Following Miles (1957, 1959), Brazier-Smith & Scott (1984), Crighton & Oswell (1991) and Lingwood & Peake (1999), I consider here a semi-infinite ($y > 0$), inviscid shear flow $U(y)$ over an infinitely long,† flexible surface, the displacement of which, $y = \eta(x, t)$, is governed by the equation of motion

$$\boldsymbol{L}\eta + m\partial_t^2\eta = -p(x, t) + f(x, t), \tag{1}$$

where $\boldsymbol{L}\eta$ is the stress resisting the displacement, m is the mass per unit area, p is the surface-motion-induced pressure, and f represents external forcing.

Brazier-Smith & Scott consider monochromatic forcing of an elastic plate of bending stiffness B (for which $\boldsymbol{L} = B\partial_x^4$) in a uniform flow and obtain a numerical description of absolute instability. Crighton & Oswell obtain analytical results for the threshold of absolute instability for Brazier-Smith & Scott's problem.

Lingwood & Peake (hereinafter referred to as LP) extend these studies to a shear flow with either a broken-line ($U = U_1 y/y_1$ in $y \leqslant y_1$, $U = U_1$ in $y \geqslant y_1$) or a Blasius boundary-layer profile. Their broken-line model permits an analytical determination of the aerodynamic force but (as they recognize) neglects the phase shift across the shear layer and predicts a decrease in threshold velocity with an increase in shear-layer thickness. Their Blasius model predicts an increase in the threshold velocity with an increase in shear-layer thickness.

† See Lucey (1998) for an extensive treatment of potential flow over panels of finite length.

The primary goal of the present investigation is an analytical description of the threshold conditions for a thin (compared with $l \equiv m/\rho$) shear layer embedded in a uniform outer flow U_1. In § 3, following Fourier transformation of the problem in § 2, I construct an approximate solution of Rayleigh's equation to obtain an analytical representation of the induced aerodynamic loading and obtain explicit results for LP's broken-line profile and for a parabolic profile.

In § 4, I show that the flow is temporally unstable for all real wavenumbers. The instability may be either Kelvin–Helmholtz-like or induced by the phase shift across the critical layer, but the latter is much weaker than the former (the situation is reminiscent of that for Charney's model of baroclinic instability of the zonal wind (Miles 1964)).

In § 5, I develop the necessary conditions for the threshold of absolute instability of an elastic plate for $0 < \varepsilon \ll 1$ through an expansion about the threshold for uniform flow ($\varepsilon = 0$). This expansion yields the threshold $\boldsymbol{F}/\boldsymbol{F}_* = 1 - 0.387\varepsilon$ for LP's broken-line profile and $\boldsymbol{F}/\boldsymbol{F}_* = 1 + \boldsymbol{C}\varepsilon^{2/3}$ ($\boldsymbol{C} > 0$) for a smooth profile with outer velocity U_1 and thickness y_1, where $\boldsymbol{F} \equiv mU_1^2 l^2/B$ is a Froude number, $\varepsilon \equiv y_1/l$, and $\boldsymbol{C}$ is a positive constant derived from conditions at the critical layer. The analysis in § 5 is extended to a membrane in the Appendix.

The present results, although in agreement with LP for their broken-line profile, contradict their conclusion that 'as $\varepsilon \to 0$ the absolute instability boundary [for a Blasius profile] is . . . a singular, and unattainable, limit . . . '. I discuss this disagreement in § 6.

2. Fourier transformation

Introducing the double Fourier-transform pair†

$$N(k,\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}(kx-\omega t)}\eta(x,t)\,\mathrm{d}x\,\mathrm{d}t \tag{2.1a}$$

and

$$\eta(x,t) = (2\pi)^{-2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(kx-\omega t)}N(k,\omega)\,\mathrm{d}k\,\mathrm{d}\omega, \tag{2.1b}$$

and similarly for p and f, and positing

$$\boldsymbol{L}N = m\omega_0^2(k)N \tag{2.2}$$

and

$$P = \rho Z(k,\omega)N, \tag{2.3}$$

where $\boldsymbol{L}$ is defined as in (1), $\omega_0(k)$ is the natural frequency for a transverse oscillation of the surface, Z is the aerodynamic impedance, and ρ is the fluid density, we transform (1) to

$$\{m[\omega_0^2(k) - \omega^2] + \rho Z(k,\omega)\}N(k,\omega) = F(k,\omega). \tag{2.4}$$

The present investigation is concerned primarily with the singularities of $N(k,\omega)$ and does not require the specification of $F(k,\omega)$.

† The paths of integration in (2.1b) typically must be deformed into the complex-k and ω planes to ensure convergence, and the corresponding limits of integration then are complex.

3. Aerodynamic impedance

The fluid motion is governed by Rayleigh's equation,

$$(U-c)\Phi''-U''\Phi=k^2(U-c)\Phi \quad ('\equiv \mathrm{d}/\mathrm{d}y), \tag{3.1}$$

where $\Phi(y)$ is the Fourier transform of the perturbation stream function. We separate the flow into: an inner domain, $0<y<y_1\ll 1/|k|$, in which

$$U(0+)=0,\quad U'(y)>0,\quad U'(y_1-)=0,\quad U''(y)<0, \tag{3.2a–d}$$

and Φ is singular at $y=y_c$ (which may have a positive-imaginary part), where

$$U(y_c)=\omega/k\equiv c; \tag{3.3}$$

an outer domain, $y>y_1$, in which $U=U_1$ and the flow is irrotational. (For the Blasius profile, which has no definite upper boundary, y_1 is a measure of the boundary-layer displacement thickness; see §5.3.) It follows from (3.2) that $0<\omega_r/|k|<U_1$ is necessary for the existence of the critical layer in the limit $\omega_i\downarrow 0$ (the subscripts r and i signify real and imaginary parts).

The linearized boundary conditions for the inner domain, which follow from the vanishing of the normal velocity at the surface and the matching to the outer solution, $\Phi\propto\exp(-\gamma y)$ in $y>y_1$, are

$$\Phi=cN \quad (y=0),\quad \Phi'+\gamma\Phi=0 \quad (y=y_1),\quad \gamma\equiv k\,\mathrm{sgn}\,k_r. \tag{3.4a–c}$$

Following Heisenberg (Drazin & Reid 1981, p. 139), we expand Φ in k^2. The first approximation, which follows from the neglect of the right-hand side of (3.1), is

$$\frac{\Phi^{(1)}}{N}=-(U-c)\left[\frac{1+\gamma(U_1-c)^2K(y)}{1+\gamma(U_1-c)^2K_0}\right], \tag{3.5}$$

where

$$K(y)=\int_y^{y_1}[U(\eta)-c]^{-2}\,\mathrm{d}\eta,\quad K_0\equiv K(0), \tag{3.6a,b}$$

and the path of integration is indented under $y=y_c$ on the hypothesis that $c_i>0$. The second approximation, which follows from the approximation of Φ by $\Phi^{(1)}$ on the right-hand side of (3.1), is

$$\Phi^{(2)}=\Phi^{(1)}+k^2(U-c)\left[\left[\frac{\Delta}{1+\Delta}\right]\left[\frac{K_0-K}{K_0}\right]Q(y_1)-Q(y)\right], \tag{3.7}$$

where

$$\Delta=\gamma(U_1-c)^2K_0,\quad \Delta_i=-\pi\gamma(U_1-c)^2(U_c''/U_c'^3) \tag{3.8a,b}$$

and

$$Q(y)=\int_0^y\frac{\mathrm{d}\eta}{[U(\eta)-c]^2}\int_\eta^{y_1}[U(\xi)-c]\Phi^{(1)}(\xi)\,\mathrm{d}\xi. \tag{3.9}$$

The Fourier transform of the perturbation surface pressure is given by

$$P/\rho=NZ=[U'\Phi-(U-c)\Phi']_{y=0}. \tag{3.10}$$

Substituting (3.7) into (3.10), we obtain

$$Z=-\gamma(U_1-c)^2(1+\Delta)^{-1}+k^2\int_0^{y_1}[\Phi^{(1)}/N]^2\,\mathrm{d}y \tag{3.11a}$$

$$=-k(U_1-c)^2[\mathrm{sgn}\,k_r-\Gamma+O(\varepsilon^2)]\quad (\varepsilon\equiv y_1/l), \tag{3.11b}$$

where

$$\Gamma = k \int_0^{y_1} \left[\left[\frac{U_1 - c}{U - c} \right]^2 - \left[\frac{U - c}{U_1 - c} \right]^2 \right] \mathrm{d}y = O(\varepsilon), \quad \Gamma_i = \Delta_i \operatorname{sgn} k_r. \qquad (3.12a,b)$$

This is equivalent to LP's (2.8), (A3) and (A4) except for a sign error (Dr Peake agrees) in their F (but this sign error does not affect their stability calculations) and their neglect of the phase shift across the critical layer.

We remark that if (3.2a) is relaxed c is replaced by $c - U_0$ in (3.4a), but U_0 enters (3.11b) only through $U(y)$.

4. Dispersion function

Substituting (3.11b) into (2.4), we obtain the dispersion function

$$D(k,\omega) \equiv F(k,\omega)/mN(k,\omega) \qquad (4.1a)$$

$$= \omega_0^2(k) - \omega^2 - (\gamma/l)(U_1 - c)^2[1 - \Gamma + O(\varepsilon^2)] \qquad (4.1b)$$

$$= (U_1/l)^2\{\sigma_0^2(\boldsymbol{k}) - \sigma^2 - \boldsymbol{k}^{-1}(\boldsymbol{k} - \sigma)^2[\operatorname{sgn} \boldsymbol{k}_r - \Gamma + O(\varepsilon^2)]\}, \qquad (4.1c)$$

where $l \equiv m/\rho$ is a characteristic length, and $\boldsymbol{k} \equiv kl$, $\sigma \equiv \omega l/U_1$, and Γ are dimensionless. $\Gamma = 0$ for a uniform flow ($y_1 \equiv 0$), and

$$\Gamma = -\frac{\varepsilon \boldsymbol{k}^2(\boldsymbol{k}^2 - \frac{8}{3}\boldsymbol{k}\sigma + 2\sigma^2)}{\sigma(\boldsymbol{k} - \sigma)^2} \qquad (4.2)$$

for LP's broken-line profile ($U = U_1 y/y_1$ in $y < y_1$), which violates (3.2c,d).

Perhaps the simplest profile that satisfies all of (3.2a–d) is the parabola

$$U/U_1 = 2\hat{y} - \hat{y}^2, \quad \hat{y} \equiv y/y_1, \qquad (4.3a,b)$$

the substitution of which into (3.12) yields

$$\Gamma = \tfrac{1}{4}\varepsilon \boldsymbol{k}(1 - \hat{c})^{1/2} \left\{ \log \left[\frac{1 + (1 - \hat{c})^{1/2}}{1 - (1 - \hat{c})^{1/2}} \right] + \mathrm{i}\pi - 2\hat{c}^{-1}(1 - \hat{c})^{1/2} \right\}$$

$$- \varepsilon \boldsymbol{k}(1 - \hat{c})^{-2}(\tfrac{8}{15} - \tfrac{4}{3}\hat{c} + \hat{c}^2), \quad \hat{c} \equiv \frac{c}{U_1} = \frac{\sigma}{\boldsymbol{k}}. \qquad (4.4a,b)$$

The flow is temporally unstable if $\sigma_i > 0$ for a zero of D. The zeros of (4.1c) for $\boldsymbol{k} > 0$ are given by

$$\sigma_\pm = \frac{\boldsymbol{k} \pm r}{1 + \boldsymbol{k}} \pm \frac{1}{2} \left[\frac{\boldsymbol{k}^2 \mp r}{1 + \boldsymbol{k}} \right]^2 \frac{\Gamma}{r} + O(\varepsilon^2), \quad r \equiv \{\boldsymbol{k}[\sigma_0^2(1 + \boldsymbol{k}) - \boldsymbol{k}^2]\}^{1/2}, \qquad (4.5a,b)$$

wherein the alternative signs are vertically ordered and $\Gamma = O(\varepsilon)$. If

$$\sigma_0^2 < \boldsymbol{k}^2/(1 + \boldsymbol{k}) \equiv \sigma_1^2, \quad r = \mathrm{i}[\boldsymbol{k}(1 + \boldsymbol{k})(\sigma_1^2 - \sigma_0^2)]^{1/2}, \qquad (4.6a,b)$$

and $\sigma_{i+} > 0$. If $\sigma_0^2 > \sigma_1^2$, $r > 0$, and $\Gamma_i > 0$ implies $\sigma_{i+} > 0$. It follows that the motion is unstable for all σ_0. But the instability for $\sigma_0^2 > \sigma_1^2$, which is induced by the phase shift across the critical layer, is much weaker than the Kelvin–Helmholtz-like instability for $\sigma_0^2 < \sigma_1^2$ (cf. Miles 1959).

$U(y)$ for the Blasius profile is available only as a series expansion or the numerical solution of a differential equation, so that it is not possible to obtain an analytical expression for Γ. However, the imaginary part of Γ, which suffices for the following

(§ 5) threshold calculation, depends only on conditions at the critical layer, and the real part of Γ may be approximated by (4.4) by approximating the Blasius profile by a parabola.

5. Absolute instability for an elastic plate

The investigation of absolute instability requires the determination of the zeros of kD, the counterpart of Crighton & Oswell's (1991) $P_+(\boldsymbol{k})$, in $\boldsymbol{k}_r > 0$ and $\sigma_i \geqslant 0$. They consider a uniform flow over a plate of bending stiffness B and areal mass density m, for which

$$\omega_0^2 = Bk^4/m, \quad \sigma_0^2 = \boldsymbol{k}^4/\boldsymbol{F}, \quad \boldsymbol{F} \equiv mU_1^2l^2/B \equiv 1/\boldsymbol{G} \qquad (5.1a\text{–}c)$$

in the present notation (the Froude number $\boldsymbol{F}$ is equivalent to their U^2). Substituting $\sigma_0^2 = \boldsymbol{G}\boldsymbol{k}^4$ into (4.1c) and multiplying by $(l/U_1)^2\boldsymbol{k}$, we obtain

$$\boldsymbol{P}(\boldsymbol{k}) = \boldsymbol{G}\boldsymbol{k}^5 - \boldsymbol{k}\sigma^2 - (\boldsymbol{k}-\sigma)^2 + \boldsymbol{P}_\varepsilon(\boldsymbol{k}) + O(\varepsilon^2) \quad (\boldsymbol{k}_r > 0), \qquad (5.2)$$

where

$$\boldsymbol{P}_\varepsilon(\boldsymbol{k}) \equiv (\boldsymbol{k}-\sigma)^2\Gamma = O(\varepsilon). \qquad (5.3)$$

$\boldsymbol{P}_\varepsilon = 0$ for a uniform flow. Crighton & Oswell show that (5.2) then admits a pinch, at which $\boldsymbol{P}(\boldsymbol{k}) = \boldsymbol{P}'(\boldsymbol{k}) = 0$, and that the threshold of absolute instability, at which $\boldsymbol{P} = \boldsymbol{P}' = \boldsymbol{P}'' = 0$, is given by

$$\boldsymbol{k}_* = 2(5/3)^{1/2} - \tfrac{5}{2} = 0.0820, \quad \sigma_* = (3/5)^{1/2}\boldsymbol{k}_* = 0.0635, \qquad (5.4a,b)$$

and

$$\boldsymbol{F}_* = 1/\boldsymbol{G}_* = 10\boldsymbol{k}_*^3 = 0.005512. \qquad (5.4c)$$

We proceed on the assumption that the pinch for a thin shear layer ($0 < \varepsilon \ll 1$) can be obtained as a small perturbation of that for a uniform flow by expanding the pinch conditions $\boldsymbol{P} = \boldsymbol{P}' = 0$ about (5.4). Introducing

$$\hat{\boldsymbol{k}} \equiv \boldsymbol{k} - \boldsymbol{k}_*, \quad \hat{\sigma} \equiv \sigma - \sigma_*, \quad \hat{\boldsymbol{G}} \equiv \boldsymbol{G} - \boldsymbol{G}_*, \qquad (5.5a\text{–}c)$$

solving $\boldsymbol{k}\boldsymbol{P}' - 5\boldsymbol{P} = 0$ for

$$\hat{\boldsymbol{k}} = \tfrac{4}{3}(1-\sigma_*)\hat{\sigma} - \mathrm{i}(A\hat{\sigma} + Q)^{1/2} + O(\hat{\sigma}^{3/2}, \varepsilon\hat{\sigma}^{1/2}), \quad A = 4\boldsymbol{k}_*\sigma_*/3, \qquad (5.6a,b)$$

and then eliminating $\hat{\boldsymbol{k}}$ from $\boldsymbol{P}' = 0$, we obtain

$$C\hat{\sigma} - 2\mathrm{i}(D\hat{\sigma} + 10\boldsymbol{k}_*^3\hat{\boldsymbol{G}} - \boldsymbol{k}_*^{-2}Q)(A\hat{\sigma} + Q)^{1/2} + 5\boldsymbol{k}_*^4\hat{\boldsymbol{G}} + 5\boldsymbol{k}_*^{-1}\boldsymbol{P}_{\varepsilon *} + O(\hat{\sigma}^2, \varepsilon\hat{\sigma}) = 0, \quad (5.7a)$$

where

$$C = 2(1-3\sigma_*), \quad D = 4\boldsymbol{k}_*^{-1}(1 - \tfrac{4}{3}\sigma_*), \quad Q \equiv \tfrac{1}{3}(\boldsymbol{k}\boldsymbol{P}_\varepsilon' - 5\boldsymbol{P}_\varepsilon)_*, \qquad (5.7b\text{–}d)$$

and $\boldsymbol{P}_{\varepsilon *}$ and Q are evaluated at $\boldsymbol{k} = \boldsymbol{k}_*$ and $\sigma = \sigma_*$. We remark that the phase shift across the critical layer implies $\boldsymbol{P}_{\varepsilon \mathrm{i}} > 0$, which is stabilizing.

5.1. *Broken-line profile*

If the phase shift across the critical layer is absent (or neglected), $\boldsymbol{P}_{\varepsilon *}$ and Q are real, and $\sigma_\mathrm{i} = 0$ requires $A\hat{\sigma} + Q = 0$ in (5.7a), which then reduces to

$$-C(Q/A) + 5\boldsymbol{k}_*^4\hat{\boldsymbol{G}} + 5\boldsymbol{k}_*^{-1}\boldsymbol{P}_{\varepsilon *} = 0. \qquad (5.8)$$

Combining (5.8) with the first-order approximation $\hat{\boldsymbol{G}} = -(\boldsymbol{F} - \boldsymbol{F}_*)/\boldsymbol{F}_*^2$, we obtain

$$\boldsymbol{F}/\boldsymbol{F}_* = 1 - \boldsymbol{k}_*^{-3}(3.136Q - 0.820\boldsymbol{P}_{\varepsilon *}) + O(\varepsilon^2). \qquad (5.9)$$

For LP's broken-line profile,

$$\boldsymbol{P}_{\varepsilon *} = -\varepsilon[\sigma^{-1}\boldsymbol{k}^2(\boldsymbol{k}^2 - \tfrac{8}{3}\boldsymbol{k}\sigma + 2\sigma^2)]_* = -0.1735\varepsilon\boldsymbol{k}_*^3, \quad Q = 0.2017\varepsilon\boldsymbol{k}_*^3, \qquad (5.10a,b)$$

the substitution of which into (5.9) yields

$$\boldsymbol{F}/\boldsymbol{F}_* = 1 - 0.775\varepsilon + O(\varepsilon^2). \qquad (5.11a)$$

This is equivalent to (cf. LP's figure 2)

$$U = \boldsymbol{F}^{1/2} = 0.0742[1 - 0.387\varepsilon + O(\varepsilon^2)] \qquad (5.11b)$$

in LP's notation and compares with

$$U = 0.0743[1 - 0.388\varepsilon + 0.125\varepsilon^2 + O(\varepsilon^3)] \qquad (5.12)$$

from their numerical data (provided by Dr Peake).

5.2. *Critical-layer-induced instability*

We now assume that $\boldsymbol{P}_{\varepsilon *}$ and Q are complex and $O(\varepsilon)$. Then $\sigma_i > 0$ requires $\hat{\sigma} = O(\varepsilon^{2/3})$ rather than $O(\varepsilon)$, and the first approximation

$$\hat{\sigma} = -(5\boldsymbol{k}_*^4/C)\hat{\boldsymbol{G}} \equiv \hat{\sigma}_1 \qquad (5.13)$$

leads to the second approximations

$$\hat{\sigma} = \hat{\sigma}_1 - (5/C\boldsymbol{k}_*)\boldsymbol{P}_\varepsilon + 2\mathrm{i}C^{-1}(D\hat{\sigma}_1 + 10\boldsymbol{k}_*^3\hat{\boldsymbol{G}})(A\hat{\sigma}_1)^{1/2} + O(\varepsilon^{4/3}). \qquad (5.14)$$

Setting $\hat{\sigma}_i = 0$, combining (5.3), (5.4), $\hat{c} = \sigma_*/\boldsymbol{k}_*$, and (3.8*b*) to obtain

$$\boldsymbol{P}_{\varepsilon i} = (\boldsymbol{k}_* - \sigma_*)^2\Delta_i = \boldsymbol{k}_*^3(1-\hat{c})^4\boldsymbol{E}, \quad \boldsymbol{E} = -\pi(U_1^2 U_c''/lU_c'^3), \qquad (5.15a,b)$$

and proceeding as above, we obtain the threshold of absolute instability

$$\boldsymbol{F} = \boldsymbol{F}_*[1 + 0.0855\boldsymbol{E}^{2/3} + O(\varepsilon^{4/3})]. \qquad (5.16)$$

For the parabolic profile (4.3), for which $y_c/y_1 = 0.525$ and $\boldsymbol{E} = 7.34\varepsilon$,

$$\boldsymbol{F} = \boldsymbol{F}_*[1 + 0.323\varepsilon^{2/3} + O(\varepsilon^{4/3})]. \qquad (5.17)$$

5.3. *Blasius profile*

The Blasius boundary-layer profile admits the expansion (Schlichting 1955, chap. VII f)

$$\frac{U}{U_1} = \sum_{n=0}^{\infty} \frac{(-)^n \alpha^{n+1} C_n}{2^n(3n+1)!}\eta^{3n+1}, \quad \alpha = 0.332, \quad C_0 = C_1 = 1, \quad C_2 = 11, \quad C_3 = 375, \quad C_4 = 27,897, \cdots, \qquad (5.18)$$

where

$$\eta = y(U_1/\nu x)^{1/2}, \quad y_1 = 5.0(\nu x/U_1)^{1/2}, \qquad (5.19a,b)$$

ν is the kinematic viscosity, x is the distance from the leading edge, and y_1 now is (by definition) the elevation at which $U/U_1 = 0.99$. The critical layer, for which $U/U_1 = \hat{c} = (3/5)^{1/2}$, lies at $\eta = 2.60$, for which $\boldsymbol{E} = 7.88\varepsilon$, and (cf. (5.17))

$$\boldsymbol{F} = \boldsymbol{F}_*[1 + 0.34\varepsilon^{2/3} + O(\varepsilon^{4/3})]. \qquad (5.20)$$

6. Conclusions

I conclude that the threshold of absolute instability for an inviscid shear flow of outer velocity U_1, local thickness y_1 and density ρ over an elastic plate of bending stiffness B and areal mass density m has the limiting ($\varepsilon \downarrow 0$) form

$$\boldsymbol{F} = \boldsymbol{F}_*(1 + \boldsymbol{C}\varepsilon^{\boldsymbol{n}}), \tag{6.1}$$

where

$$\varepsilon = y_1/l, \quad l \equiv m/\rho, \quad \text{and} \quad \boldsymbol{F} = mU_1^2 l^2/B \tag{6.2a–c}$$

is a Froude number; $\boldsymbol{F}_* = 0.005512$ is the limiting value of $\boldsymbol{F}$ for a uniform flow ($\varepsilon = 0$). $\boldsymbol{C} < 0$ and $\boldsymbol{n} = 1$ in the absence of a critical-layer phase shift; in particular, $\boldsymbol{C} = -0.775$ for Lingwood & Peake's (1999) broken-line profile. $\boldsymbol{C} > 0$ and $\boldsymbol{n} = 2/3$ for a flow with a critical-layer phase shift; in particular, $\boldsymbol{C} = 0.332$ for a parabolic profile, and $\boldsymbol{C} = 0.34$ for the Blasius boundary-layer profile (for which y_1 is the elevation at which $U = 0.99U_1$).

This last result contradicts LP's conclusion that 'as $\varepsilon \to 0$ the absolute instability boundary [for a Blasius profile] is... a singular, and unattainable, limit...'. I do not know the source of this disagreement. The principal differences between their solution and the present solution are that they do not restrict ε to be small and their solution is numerical. I recognize that the assumption $0 < \varepsilon \ll 1$, on which the present analysis rests, precludes the global analysis that is required for a complete determination of the pinch, but Crighton & Oswell (1991) have provided this analysis for $\varepsilon = 0$, and I argue (following the suggestion of an anonymous referee) that their result provides a basis for a small-perturbation continuation into $0 < \varepsilon \ll 1$. This argument is supported by the agreement between LP's numerical solution and the present analytical solution for the broken-line profile.

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Appendix. Membrane

The preceding analysis may be extended to any flexible boundary for which $\omega_0^2 \propto k^{2n}$. Perhaps the simplest, but still representative, example is a membrane stretched by the uniform tension T (in the x-direction), for which $\omega_0^2 = Tk^2/m \equiv c_0^2 k^2$ and

$$\sigma_0^2 = \boldsymbol{k}^2/\boldsymbol{F}, \quad \boldsymbol{F} \equiv U_1^2/c_0^2 \equiv 1/\boldsymbol{G}, \tag{A 1a,b}$$

where c_0 is the wave speed (which is constant for a membrane), and $\boldsymbol{F}$ is a Froude number. The counterparts of (5.2) and (5.4) are

$$\boldsymbol{P}(\boldsymbol{k}) = \boldsymbol{G}\boldsymbol{k}^3 - \boldsymbol{k}\sigma^2 - (\boldsymbol{k} - \sigma)^2 + \boldsymbol{P}_\varepsilon(\boldsymbol{k}) + O(\varepsilon^2), \tag{A 2}$$

$$\boldsymbol{k}_* = 2\sqrt{3} - 3 = 0.464, \quad \sigma_* = 2 - \sqrt{3} = 0.268, \quad \boldsymbol{F}_* = 1/\boldsymbol{G}_* = 3\boldsymbol{k}_* = 1.393. \tag{A 3a–c}$$

The elimination of $\hat{\boldsymbol{k}} \equiv \boldsymbol{k} - \boldsymbol{k}_*$ between $\boldsymbol{P}(\boldsymbol{k}) = \boldsymbol{P}'(\boldsymbol{k}) = 0$ yields

$$\hat{\boldsymbol{k}} = 2(1 - \sigma_*)\hat{\sigma} - \mathrm{i}(A\hat{\sigma} + Q)^{1/2} + O(\hat{\sigma}^{3/2}), \quad A = 2\boldsymbol{k}_*\sigma_*, \tag{A 4a,b}$$

and

$$C\hat{\sigma} - 2\mathrm{i}[2(1 - \sigma_*)\boldsymbol{k}_*^{-1}\hat{\sigma} + 3\boldsymbol{k}_*\hat{\boldsymbol{G}}](A\hat{\sigma} + Q)^{1/2} + 3\boldsymbol{k}_*^2\hat{\boldsymbol{G}} + 3\boldsymbol{k}_*^{-1}\boldsymbol{P}_\varepsilon = 0 \tag{A 5a}$$

where

$$C = 2(1 - 2\sigma_*), \quad Q = \boldsymbol{k}\boldsymbol{P}'_\varepsilon - 3\boldsymbol{P}_\varepsilon, \tag{A 5b,c}$$

in place of (5.6) and (5.7). Proceeding as in § 5.2, we obtain

$$\hat{\sigma} = \hat{\sigma}_1 - 3C^{-1}\boldsymbol{k}_*^{-1}\boldsymbol{P}_\varepsilon + 2\mathrm{i}C^{-1}\boldsymbol{k}_*(2\hat{\sigma}_1/\sqrt{3})^{3/2}, \quad \hat{\sigma}_1 = -3C^{-1}\boldsymbol{k}_*^2\hat{\boldsymbol{G}}, \qquad (\mathrm{A}\,6a, b)$$

and

$$\boldsymbol{F}/\boldsymbol{F}_* = 1 + 1.053\boldsymbol{k}_*^{-7/3}\boldsymbol{P}_{\varepsilon i}^{2/3}, \qquad (\mathrm{A}\,7)$$

which reduces, through (5.15), to

$$\boldsymbol{F} = \boldsymbol{F}_*[1 + 0.0965\varepsilon^{2/3} + O(\varepsilon^{4/3})] \qquad (\mathrm{A}\,8)$$

for the parabolic profile.

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